Integral representability of linear operators on Orlicz spaces

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Let L^{Φ} be an Orlicz space (over a finite atomless measure space (Ω, Σ, μ)) defined by a finite valued Orlicz function (not necessarily convex), and let Φ^* denote the complementary function of Φ in the sense of Young.

For $\varepsilon > 0$ let $U_{\Phi}(\varepsilon) = \left\{u \in L^{\Phi} : m_{\Phi}(u) \leq \varepsilon\right\}$,where $m_{\Phi}(u) = \int\limits_{\Omega} \Phi(|u(t)|)\, d\mu$. Then the family of all sets of the form:

$$\bigcup_{n=1}^{\infty} \left(\sum_{i=1}^{n} U_{\Phi}(\varepsilon_i) \right),$$

where (ε_n) is a sequence of positive numbers, constitutes a base of neighbourhoods of zero for some linear topology $T_{\Phi}^{\wedge}$ on L^{Φ}, which is called the modular topology (see [N]). Then $T_{\Phi}^{\wedge}$ is the finest Lebesgue'a topology on L^{Φ}.

Let $(X, \|\cdot\|_X)$ be a Banach space and $T : L^{\Phi} \longrightarrow X$ be a linear operator. Let us set:

$$P_{\Phi}(T) = \sup\left\{ \sum_{i=1}^{n} \left\|\alpha_i T\left(\chi_{A_i}\right)\right\|_X : s = \sum_{i=1}^{n} \alpha_i \chi_{A_i} \in \mathbb{S}(\mu), m_{\Phi}(s) \leq 1 \right\}.$$

We prove the following:

Theorem. Assume that X has the Radon-Nikodym property (with respect to μ). Then for a linear operator $T : L^{\Phi} \longrightarrow X$ the following conditions are equivalent:

(i)$P_{\Phi}(T) < \infty$ and T is order continuous (i. e. $\|T(u_n)\|_X \longrightarrow 0$ as $u_n \xrightarrow{(o)} 0$ in L^{Φ})

(ii) There exists a unique $g \in L^{\Phi^*}(X)$ (= the Orlicz-Bochner space defined by Φ^*) such that

$$T(u) = T_g(u) = \int\limits_{\Omega} u(\omega)\, g(\omega)\, d\mu \quad \text{for all } u \in L^{\Phi}.$$

Moreover, for $g \in L^{\Phi^*}(X)$ we have:

$$P_{\Phi}(T_g) = \sup\left\{ \left| \int\limits_{\Omega} u(\omega) \|g(\omega)\|_X \, d\mu \right| : u \in L^{\Phi}, m_{\Phi}(u) \leq 1 \right\}.$$

(iii)$P_{\Phi}(T) < \infty$ and T is $(T_{\Phi}^{\wedge}, \|\cdot\|_X)$-continuous.

In case when Φ is convex and Φ satisfies the Δ_2 -condition, the equivalence (i)$\Leftrightarrow$(ii) was obtained by J. Uhl (see [U]).

References:

[N] M. Nowak, *On the modular topology on Orlicz spaces*, Bull. Polish Acad. Sci. Math., 36 (1988), no. 9-10, 553-562.

[U] J.J.Uhl, *On class of operators on Orlicz spaces*,Studia Math., 40 (1971) , pp. 17-22