Weighted Hardy spaces on an interval and Jacobi series

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In the papers Chang–Krantz–Stein [2] and Chang–Dafni–Stein [1], the authors introduced the function spaces $h^p_z(\bar{\Omega})$ and $h^p_d(\bar{\Omega})$ for bounded domain $\Omega \subset \mathbf{R}^n$ with smooth boundary and gave estimates for solutions to the Laplace equation. The essence of the latter estimates is the boundedness of certain singular integral operators in their spaces. In this talk I would like to give a different approach to define H^p spaces on a domain of $\mathbf{R}^n$ and show that certain singular integral operators act as bounded operators in those spaces. To be precise, although the results for the H^p spaces will be sufficiently general, the results for singular integral operators will be restricted to a special one dimensional case. The results, however, will give interesting suggestions for generalizations to higher dimensional case.

To define the H^p spaces on a domain Ω of $\mathbf{R}^n$, we use the same maximal function f^* introduced in [3]. If λ is a Borel measure on Ω that satisfies the doubling condition in Ω, then we define $H^p(\Omega, \lambda)$ to be the set of all distributions f on Ω for which f^* belongs to $L^p(\Omega, \lambda)$.

Two important properties of the space $H^p(\Omega, \lambda)$ are the following. First, if $\Phi : \Omega \to \tilde{\Omega}$ is a C^∞ diffeomorphism that satisfies certain conditions, then the change of variables $f(x) = \tilde{f}(\Phi(x))$ (in the sense of distribution) gives an isomorphism between $H^p(\Omega, \lambda)$ and $H^p(\tilde{\Omega}, \lambda_*)$, where λ_* is the image measure of λ under the map Φ. It can be shown that if the dimension is 2 then every conformal mapping between arbitrary open subsets of $\mathbf{R}^2 = \mathbf{C}$ satisfies the required conditions. Secondly, if w is a C^∞ positive function that satisfies the estimates $|\partial^\alpha w(x)| \leqq A_\alpha w(x) \mathrm{dis}\,(x, \Omega^c)^{-|\alpha|}$, then the mapping $f \mapsto wf$ gives an isomorphism from $H^p(\Omega, w^p\lambda)$ onto $H^p(\Omega, \lambda)$.

The space $H^p(\Omega, \lambda)$ can be characterized by atomic decomposition. By using the above two properties, we see that we can use the atoms that satisfy the moment condition of the form $\int w(x)P(\Phi(x))f(x)dx = 0$, where $P(y)$ are polynomials and Φ and w are as mentioned above. This modified atom is useful in the application given below.

As an application of the space $H^p(\Omega, \lambda)$, we use the special case that $\Omega = (0, \pi)$ (open interval of $\mathbf{R}$) and $d\lambda(\theta) = \theta^{ap}(\pi - \theta)^{bp} d\theta$ and consider the transplantation operators between two Jacobi series; this is the main subject of the talk. We show that Muckenhoupt's theorem in [5], that gives weighted L^p boundedness of the transplantation operators, are generalized to the case $p \leqq 1$. Details of the last results will appear in [4].

References

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