Metric entropy in function spaces

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The notion of metric entropy is known since more than fifty years, and it has found a lot of applications in many different areas of mathematics. In the context of operators there is the following version of this notion:

The k^{th} *dyadic entropy number* of a of bounded linear operator $T : X \to Y$ between quasi-Banach spaces is defined as

$$e_k(T) = \inf\{\varepsilon > 0 : T(B_X) \text{ can be covered by } 2^{k-1} \text{ balls in } Y \text{ of raduis } \varepsilon\}\,,$$

where B_X denotes the closed unit ball in X. Obviously, T is compact if and only if $\lim_{k\to\infty} e_k(T) = 0$, therefore the asymptotic behaviour of the sequence $(e_k(T))$ describes in a certain sense the 'degree' of compactness of T.

Entropy numbers are closely related to eigenvalues via the Carl-Triebel inequality; this is the basis for applications to spectral theory. In recent years there was an increasing interest in entropy numbers of embeddings of function spaces.

Here we mainly concentrate on embeddings of *weighted Besov spaces*

$$B^s_{p,q}(\mathbb{R}^d, w) := \{f \in \mathcal{S}'(\mathbb{R}^d) : fw \in B^s_{p,q}(\mathbb{R}^d)\}\,,$$

where $B^s_{p,q}(\mathbb{R}^d)$ are the usual Besov spaces, with $0 < p, q \leq \infty$ and $s \in \mathbb{R}$. Under certain regularity conditions on the weights, we give necessary and sufficient conditions such that there is a compact embedding

$$B^{s_1}_{p_1,\,q_1}(\mathbb{R}^d, w_1) \hookrightarrow B^{s_2}_{p_2,\,q_2}(\mathbb{R}^d, w_2)\,.$$

Moreover, we establish asymptotically sharp bounds for the entropy numbers of such embeddings for several (large) classes of weights. We explain the strategy and main ideas of the proofs, and discuss some new phenomena which do not occur in the unweighted case.

The talk is based on a recently published series of joint papers with Hans-Gerd Leopold (Jena), Winfried Sickel (Jena) and Leszek Skrzypczak (Poznań).