Approximative compactness and continuity of metric projection in Banach spaces

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If $(X, \|\cdot\|)$ is a Banach space and A is nonempty subset of X we define the metric projection P_A from X onto A by

$$P_A(x) = \{y \in X : \ \|x - y\| = dist(x, A)\},$$

where $dist(x, A) = inf\{\|x - z\| : \ z \in A\}$. It is known that if X is approximatively compact (which means that any nonempty closed and convex set A in X is approximatively compact, that is for any $\{x_n\} \subset A$ and any $y \in X$ such that $\|x_n - y\| \to dist(y, A)$ we have that $\{x_n\}$ is a Cauchy subsequence), then for any nonempty closed and convex set A in X, $P_A(x) \neq \emptyset$ for any $x \in X$ and P_A continuous on X.

We prove that if X is midpoint locally uniformly rotund, then for any nonempty closed and conex set A in X, approximative compactness is also necessary for continuity of the metric projection $P_A(\cdot)$ on X (thanks midpoint local uniform rotundity, $P_A(x)$ is a singleton for any $x \in X$).

The result presented here is from my joint paper with H. Hudzik, Y. Wang and M. Wisła.