Convexity, compactness and distances

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Our starting point is the following extension of the Krein-Šmulian Theorem: if X is a Banach space, $Z \subset X$ a convex subset of X and $K \subset X^{**}$ a w*−compact subset, then $d(\overline{\text{co}}^{w^*}(K), Z) \leq 5d(K, Z)$ and, if $Z \cap K$ is w*-dense in K, then $d(\overline{\text{co}}^{w^*}(K), Z) \leq 2d(K, Z)$. Moreover, there exist a Banach space X and two w*-compact subsets $K_1, K_2 \subset X^{**}$ such that: (1) $K_1 \cap X$ is w*-dense in K_1, $d(K_1, X) = \frac{1}{2}$ and $d(\overline{\text{co}}^{w^*}(K_1), X) = 1$; (ii) $d(K_2, X) = \frac{1}{3}$ and $d(\overline{\text{co}}^{w^*}(K_2), X) = 1$. So, the best universal constant M of the extension of the Krein-Šmulian Theorem satisfies $3 \leq M \leq 5$. For the category of w*-compact subsets $K \subset X^{**}$ such that $X \cap K$ is w*-dense in K, the best constant M is exactly $M = 2$.

In a dual Banach space X^* it can be studied when the distances $d(\overline{\text{co}}^{w^*}(K), Y)$ are M-controlled by the distances $d(K, Y)$ (that is, if $d(\overline{\text{co}}^{w^*}(K), Y) \leq Md(K, Y)$ for some $1 \leq M < \infty$), Y being a subspace of a dual Banach space X^* and K a w*-compact subset of X^*. We characterize the Banach spaces Y that have universally control (that is, Y has control (in fact 3-control) inside every dual Banach space X^* that contains Y as a subspace) and the class of w*compact subsets $K \subset X^*$ such that $\overline{\text{co}}(K) = \overline{\text{co}}^{w^*}(K)$.

The dual space $\ell_\infty(H)$ is interesting in order to to investigate the control of some of its subspaces Z. We study some special cases, namely: (1) $Z = C^*(H)$ when H is a topological space (metrizable, first axiom, compact, etc.); (2) Z is the subspace $B_{1b}(H)$ of bounded functions of the first Baire class on H, H being a metric space.

References

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