Weak compactness and the structure of Banach spaces

Marián Fabian
Mathematical Institute of Czech Academy of Sciences

A Banach space is called weakly compactly generated (WCG) if it contains a weakly compact set whose linear hull is dense in it. Originating from now standard results on WCG spaces obtained by Amir and Lindenstrauss in late sixties, we characterize several relatives of the class of WCG spaces.

Theorem1. A Banach space X is a subspace of a WCG space, if and only if for every $\varepsilon > 0$ its unit ball B_X can be split into countably many pieces, each piece being $\varepsilon-$weakly compact, if and only if X is weakly Lindelöf determined and moreover for every $\varepsilon > 0$ the ball B_X can be split into countably many pieces, each piece being an $\varepsilon-$Asplund set, if and only if the dual unit ball B_{X^*} is an Eberlein compact.

Theorem 2. X is a subspace of a Hilbert generated space, if and only if it admits an equivalent uniformly Gâteaux differentiable norm, if and only if B_{X^*} is uniform Eberlein compact.

As consequences, we get functional analytic proofs of several topological results: A continuous image of a (uniform) Eberlein compact is (uniform) Eberlein (Benyamini at al.). A compact space is Eberlein if and only if it is both Corson and quasi-Radon-Nikodým (Arvanitakis). We also get a criterion of Talagrand-Argyros-Farmaki for recognizing Eberlein compacta sitting in the $\Sigma-$product of real lines.

The presented results have been obtained by a group of collaborators: G. Godefroy, P. Hájek, V. Montesinos, V. Zizler and the lecturer.